Theorem: 1

In any triangle, the sides are proportional to the sines of the opposite angles.

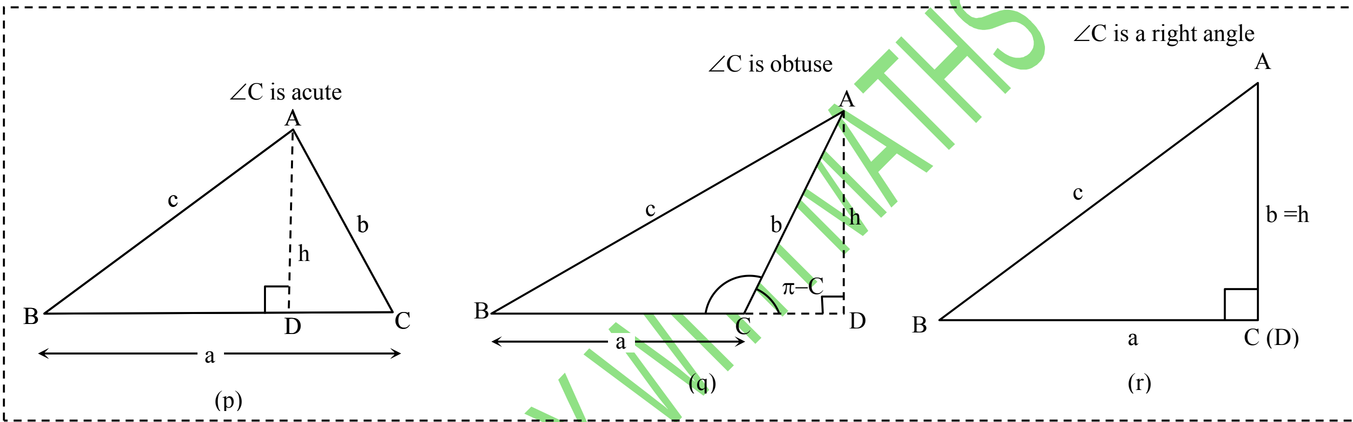
i.e. in $\Delta ABC, \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Proof:

Let ΔABC be any triangle.

Since sum of all the angles of a triangles is 180°, each angle of a triangle cannot be obtuse.

Then, consider the following three cases:



Draw AD⊥ BC or BC produced.

Let AD=h.

Then, we have $\dfrac{AD}{AB} = \sin B$

$\therefore \dfrac{h}{c} = \sin B$

$\therefore h = c \sin B$

From fig(p), $\dfrac{AD}{AC} = \sin C$

$\therefore \dfrac{h}{b} = \sin C \;\therefore h = b \sin\, C$

From fig (q), $\frac{AD}{AC} = \sin(\pi - C)$

$\therefore \frac{h}{b} = \sin C \qquad \therefore h = b\sin C$

From fig (r), $\frac{AD}{AC} = 1 = \sin\frac{\pi}{2} = \sin C$

$\therefore \frac{h}{b} = \sin C$

$\therefore h = b\sin C$

Thus, is each case, we have h =b sinC …………(ii)

From (i) and (ii), we get

C sinB= b sin C

$\therefore \frac{b}{\sin B} = \frac{c}{\sin C}$

Similarly we can prove, $\frac{b}{\sin B} = \frac{a}{\sin A}$

Hence, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

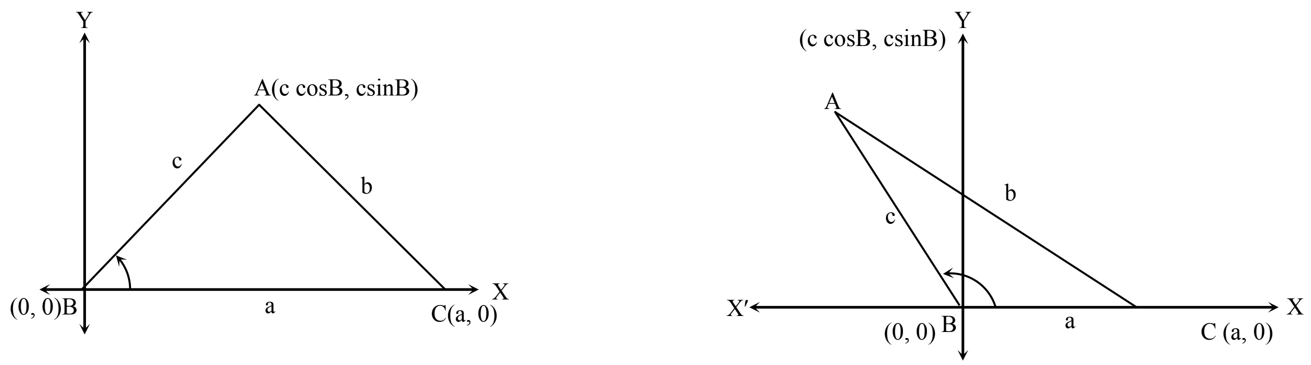
Theorem:2

In any ΔABC, prove that

i. $a^2 = b^2 + c^2 - 2bc\cos A$ **ii.** $b^2 = c^2 + a^2 - 2ca\cos B$ **iii.** $c^2 = a^2 + b^2 - 2ab\cos C$

Proof:

Consider that for ΔABC, ∠B is in a standard position i.e. vertex B is at the origin and the side BC is along positive X-axis. As ∠B is an angle of a triangle, ∠B can be acute or ∠B can be obtuse.



Using the Cartesian co-ordinate system, we get $B \equiv (0,0), A \equiv (c\cos B, c\sin B)\ and\ C \equiv (a,0)$

Now consider $l(CA) = b$

$\therefore b^2 = (a - c\cos B)^2 + (0 - c\sin B)^2$ ....(by distance formula)

$= a^2 - 2ac\cos B + c^2\cos^2 B + c^2\sin^2 B$

$= a^2 - 2ac\cos B + c^2(\sin^2 B + \cos^2 B)$

$= a^2 - 2ac\cos B + c^2$

$\therefore b^2 = a^2 + c^2 - 2ac\cos B$

Similarly, $a^2 = b^2 + c^2 - 2bc\cos A\ and\ c^2 = a^2 + b^2 - 2ab\cos C$ can be proved.

Theorem:3

In any ΔABC, prove that

(i) $a = b\cos c + c\cos B$ **(ii)** $b = c\cos A + a\cos C$ **(iii)** $c = a\cos B + b\cos A$

$\therefore \cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$

Similarly $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

For proving (iii), consider R.H. S of (iii)

R.H. S $= a\cos B + b\cos A$

$= a.\dfrac{c^2 + a^2 - b^2}{2ac} + b.\dfrac{b^2 + c^2 - a^2}{2bc}$

$= \dfrac{c^2 + a^2 - b^2}{2c} + \dfrac{b^2 + c^2 - a^2}{2c}$

$= \dfrac{c^2 + a^2 - b^2 + b^2 + c^2 - a^2}{2c}$

$= \dfrac{2c^2}{2c}$

$= c$

$\therefore$ L.H. S. =L.H.S

$\therefore c = a\cos B + b\cos A$

Similarly, $a = b\cos C + c\cos B \; and \; b = c\cos A + a\cos C$ can be prove by using cosine rule.

Theorem :4

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$ (iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

Proof:

We know that, $1-\cos A = 2\sin^2\frac{A}{2}$

$$\therefore 1-\left(\frac{b^2+c^2-a^2}{2bc}\right) = 2\sin^2\frac{A}{2} \qquad \text{.......[By cosine rule]}$$

$$\therefore \frac{2bc-b^2-c^2+a^2}{2bc} = 2\sin^2\frac{A}{2}$$

$$\therefore \frac{a^2-(b^2-2bc+c^2)}{2bc} = 2\sin^2\frac{A}{2}$$

$$\therefore \frac{a^2-(b-c)^2}{2bc} = 2\sin^2\frac{A}{2}$$

$$\therefore \frac{(a+b-c)(a-b+c)}{2bc} = 2\sin^2\frac{A}{2}$$

$$\therefore \frac{(a+b+c-2c)(a+b+c-2b)}{2bc} = 2\sin^2\frac{A}{2}$$

$$\therefore \frac{(2s-2c)(2s-2b)}{2bc} = 2\sin^2\frac{A}{2} \qquad \text{........}\left[\because a+b+=2s\,(given)\right]$$

$$\therefore \frac{(s-c)(s-b)}{bc} = \sin^2\frac{A}{2}$$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \text{.....}\left[As\angle\frac{A}{2} \, is \, an \, acute \, angle \therefore \sin\frac{A}{2} > 0\right]$$

Similarly, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}} \; and \; \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ can be proved.

Theorem :5

In any ΔABC, if $a+b+c=2s$, then

(i) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$ (ii) $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

Proof:

We know that, $1+\cos A=2\cos^2\frac{A}{2}$

$\therefore 2\cos^2\frac{A}{2}=1+\frac{b^2+c^2-a^2}{2bc}$ ...... [By cosine rule]

$=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}=\frac{(b+c+a)(b+c-a)}{2bc}=\frac{(a+b+c)(a+b+c-2a)}{2bc}$

$\therefore 2\cos^2\frac{A}{2}=\frac{2s(2s-2a)}{2bc}$ ......$[\because a+b+c=2s(given)]$

$\therefore \cos^2\frac{A}{2}=\frac{s(s-a)}{bc}$

$\therefore \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ ......$\left[As\ \frac{A}{2}\ is\ an\ acute\ angle \therefore \cos\frac{A}{2}>0\right]$

Similarly, $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$ *and* $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$ can be proved.

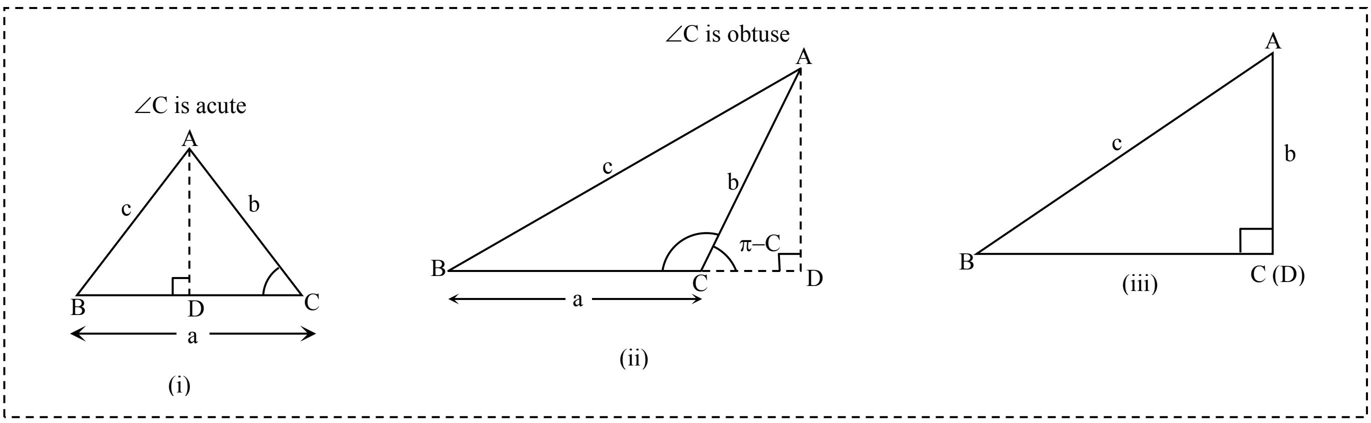
Theorem:6

Prove that with the usual notations, the area of a ΔABC is given by

$A(\Delta ABC)=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$

Proof:

We consider the following three cases:



Draw AD ⊥ BC or BC produced.

By definition of area of triangle, $A(\Delta ABC)=\frac{1}{2}\times BC\times AD$ ....(i)

From fig(i), $\frac{AD}{AC}=\sin C$

$\therefore AD=b\sin C$

From Fig (ii), $\frac{AD}{AC}=\sin(\pi-C)$

$\therefore AD=b\sin C$

From fig(iii), $\frac{AD}{AC}=1=\sin\frac{\pi}{2}$

$\therefore AD=b\sin C$

Thus, in each case, we have AD=b sin C

$\therefore A(\Delta ABC)=\frac{1}{2}\times a\times b\sin C$ ....[From (i)]

$=\frac{1}{2}ab\sin C$

Similarly, $A(\Delta ABC)=\frac{1}{2}bc\sin A=\frac{1}{2}ac\sin B$ can be proved

Napier' s analogies

Theorem:7

In any ΔABC, prove that

(i) $\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$ **(ii)** $\tan\left(\frac{C-A}{2}\right)=\left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

(iii) $\tan\left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$

Proof:

In ΔABC by sine rule, we have

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

$\therefore a = k\sin a, \text{b} = k\sin B \; and \; c = k\sin\text{C}$

Now, consider $\dfrac{a-b}{a+b} = \dfrac{k\sin A - k\sin B}{k\sin A + k\sin B}$

$$= \frac{\text{sinA} - \text{sinB}}{\sin A + \sin B}$$

$$= \frac{2\cos\left(\frac{A+B}{2}\right).\sin\left(\frac{A-A}{2}\right)}{2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}$$

$$= \cot\left(\frac{A+B}{2}\right).\tan\left(\frac{A-B}{2}\right)$$

$$= \cot\left(\frac{\pi}{2} - \frac{C}{2}\right).\tan\left(\frac{A-B}{2}\right) \qquad ....[\because A+B+C=\pi]$$

$$= \tan\frac{C}{2}.\tan\left(\frac{A-B}{2}\right)$$

$$\therefore \frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$$

$$\therefore \tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$$

Similarly, $\tan\left(\dfrac{B-C}{2}\right) = \left(\dfrac{b-c}{b+c}\right)\cot\dfrac{A}{2} \; and \; \tan\left(\dfrac{C-A}{2}\right) = \left(\dfrac{c-a}{c+a}\right)\cot\dfrac{B}{2}$ can be proved.

Theorem :8

The combined equation of a pair of a straight lines passing through the origin is a homogeneous equation of degree 2 in x and y.

Proof:

Let $a_1x + b_1y = 0 \; and \; a_2x + b_2y = 0$ be a pair of lines passing through the origin.

$\therefore$ their joint equation is $(a_1x + b_1y) = 0$

$$\therefore a_1a_2x^2 + a_1b_2xy + b_1a_2xy + b_1b_2y^2 = 0$$

$$\therefore (a_1a_2)x^2 + (a_1b_2 + a_2b_1)xy + (b_1b_2)y^2 = 0$$

This is a homogeneous equation of degree two in $x\,and\,y$.

Let $a_1a_2 = a, a_1b_2 + a_2b_1 = 2h, b_1b_2 = b,$ then the above equation becomes $ax^2 + 2hxy + by^2 = 0$

Theorem :9

A homogenous equation of degree 2 in x and y

i.e; $ax^2 + 2hxy + by^2 = 0$ **represented a pair of lines through the origin if** $h^2 - ab \geq 0$**.**

Proof:

Let $ax^2 + 2hxy + by^2 = 0$ ....(i)

Be a homogenous equation of degree 2 in x and y.

Case I:

If $b = 0(i.e, a \neq 0, h \neq 0),$ then the equation (i) reduces to $ax^2 + 2hxy = 0$

i.e $x(ax + 2hy) = 0$

This represents two lines, $x = 0\,and\,ax + 2hy = 0,$ both passing through the origin.

Case-II

If $a = 0\,and\,b = 0(i.e, h \neq 0)$, then the equation (i) reduces to $2hxy = 0, i.e., \text{xy} = 0$ which represents the coordinates axes and they pass through the origin.

Cash –III

If $b \neq 0,$

Multiplying both sides of equation (i) by b, we get $abx^2 + 2hbxy + b^2y^2 = 0$

$\therefore b^2y^2 + 2hbxy = -abx^2$

To make L.H.S a complete square, we add h^2x^2 on both the sides.

$\therefore b^2y^2 + 2hbxy + h^2x^2 = -abx^2 + h^2x^2$

$\therefore (by + hx)^2 = (h^2 - ab)x^2 \therefore (by + hx)^2 = \left[\left(\sqrt{h^2 - ab}\right)x\right]^2$

$\therefore (by + hx)^2 - \left[\left(\sqrt{h^2 - ab}\right)x\right]^2 = 0$

Theorem:10

If θ is the measure of acute angle between the pair of lines given by $ax^2+2hxy+by^2=0$, **then prove that** $\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|, a+b\neq 0$

Proof:

The given combined equation of lines is $ax^2+2hxy+by^2=0$

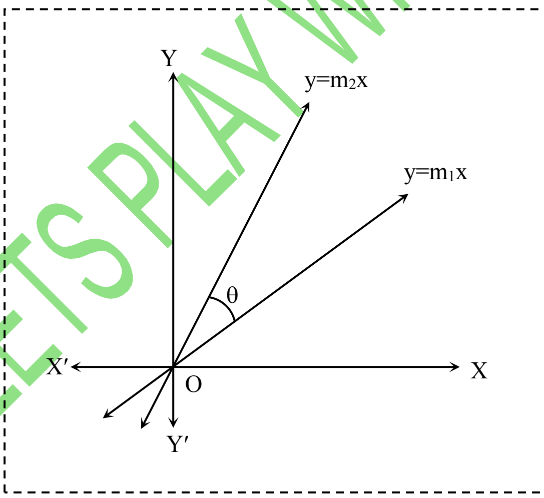
Let m_1 and m_2 be the slopes of the lines represented by $ax^2+2hxy+by^2=0$

$\therefore m_1+m_2=-\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b}, b\neq 0$

If $\frac{a}{b}=-1$, then $m_1.m_2=-1$

$\therefore$ lines are perpendicular.

So we assume that $\frac{a}{b}\neq -1$



Now, $(m_1-m_2)^2=(m_1+m_2)^2-4m_1.m_2$

$=\left(-\frac{2h}{b}\right)^2-\frac{4a}{b}$

$=\frac{4h^2}{b^2}-\frac{4a}{b}$

$\therefore (m_1-m_2)^2=\frac{4h^2-4ab}{b^2}=\frac{4(h^2-ab)}{b^2}$

Taking square root on both the sides, we get $m_1 - m_2 = \pm\frac{2\sqrt{h^2 - ab}}{b}$

Let θ be the acute angle between the lines. $\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1.m_2}\right| = \left|\frac{\frac{\pm 2\sqrt{h^2-ab}}{b}}{1+\frac{a}{b}}\right|, \frac{a}{b} \neq -1$

$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|, a+b \neq 0$

Remarks:

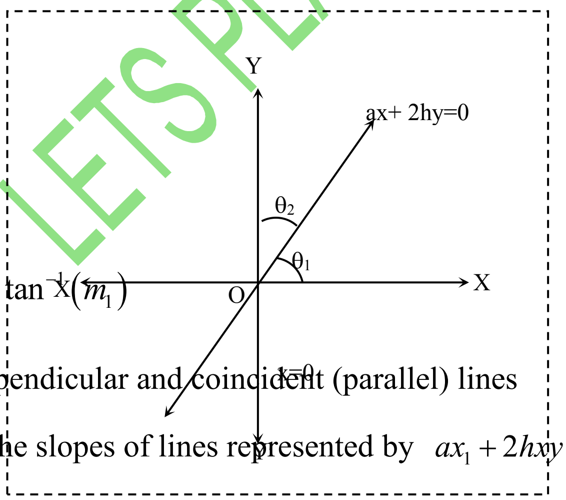
If $b = 0$, then $ax^2 + 2hxy + by^2 = 0$ represents two lines whose equations are $x = 0$ and $ax + 2hy = 0$.

Let θ_1 be the inclination of $ax + 2hy = 0$ with X-axis.

$\therefore m_1 = \tan\theta_1$

$\therefore \theta_1 = \tan^{-1}(m_1)$

Let θ be the angle between $x = 0$ and $ax + 2hy = 0$.



$\therefore \theta = \frac{\pi}{2} - \theta_1 = \frac{\pi}{2} - \tan^{-1}(m_1)$

Conditions for perpendicular and coincident (parallel) lines

Let m1 and m2 be the slopes of lines represented by $ax_1 + 2hxy + by^2 = 0$

(i) Lines are perpendicular to each other, if and only if $m_1.m_2 = -1$

i.e if $\frac{a}{b} = -1$

$i.e, if\ a = -b$

i.e. If $a + b = 0$

$\therefore$ Lines are perpendicular to each other if and only if $a + b = 0$

(ii) Lines are coincident (parallel), if and only if $m_1 = m_2$

i.e., If $m_1 - m_2 = 0$

i.e., If $\pm\frac{2\sqrt{h^2 - ab}}{b} = 0$

i.e, If $h^2 - ab = 0$

i.e. if $h^2 = ab$

$\therefore$ Lines are coincident if an only if $h^2 = ab$.

Theorem: 11

Two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exist scalars m and n, at least one of them is non-zero such that $m\vec{a} + n\vec{b} = \vec{0}$

Proof:

Suppose $\vec{a}$ and $\vec{b}$ are collinear .

$\therefore$ There exists a scalar $t \neq 0$ such that $\vec{a} = t\vec{b}$

$\therefore \vec{a} - t\vec{b} = \vec{0}$

$\therefore m\vec{a} + n\vec{b} = \vec{0}$, where $\mathrm{m} = 1$ and $\mathrm{n} = -1$

Conversely, suppose $m\vec{a} + n\vec{b} = \vec{0}$ *and* $m \neq 0$, then $\vec{a} = \left(-\frac{n}{m}\right)\vec{b} = t\vec{b}$, *where* $t = -\frac{n}{m}$ is a scalar.

$\therefore \vec{a}$ and $\vec{b}$ are collinear

Linear combination of vectors:

If $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are n vectors and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n scalar then the vector

Linear $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are n vectors and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n scalar then the vector

$\vec{a} = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \ldots + \alpha_n\vec{a}_n$ is called the linear combination of the vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$. The linear combination

$\vec{a} = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \ldots + \alpha_n\vec{a}_n$ is called non-zero linear combination, if at least one of the scalar $\alpha_1, \alpha_2, \ldots, \alpha_n$ is non zero

Coplanar Vectors

A set of vectors is coplanar, if their corresponding position vectors lie in the same plane.

Theorem :12

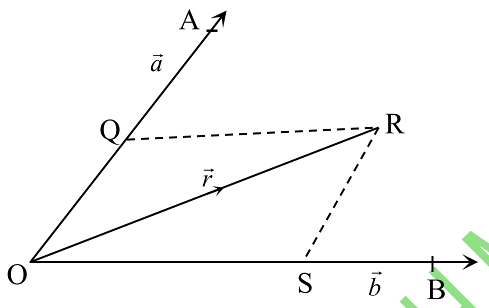
Let $\vec{a}\ and\ \vec{b}$ be a non-collinear vectors. A vector $\vec{r}$ is coplanar with $\vec{a}\ and\ \vec{b}$ if and only if their exist unique scalar t_1 and t_2 such that $\vec{r}=t_1\vec{a}+t_2\vec{b}$.

Proof:

Suppose $\vec{a},\vec{b}\ and\ \vec{r}$ are coplanar.

$\therefore \vec{a}=\overline{OA},\mathrm{\vec{b}}=\overline{\mathrm{OB}}\ and\ \vec{r}=\overline{OR}$ are in the same plane, where O is the origin.

Suppose $\vec{r}$ is collinear with either of them, say $\vec{a}$, then $\vec{r}=t_1\vec{a}$, where t_1 is a scalar



$\therefore\ \vec{r}=t_1\vec{a}+t_2\vec{b}\ (where\ t_2=0)$

Now, let $\vec{r}$ be non-collinear with $\vec{a}\ and\ \vec{b}$

Let Q and S lie on the vectors $\vec{a}\ and\ \vec{b}$ respectively such that OQRS is a parallelogram.

Now the vectors $\overline{OQ}\ and\ \vec{a}$ are collinear and similarly, the vectors $\overline{OS}\ and\ \vec{b}$ are collinear.

$\therefore$ There exist scalar t_1 and t_2 such that $t_1\vec{a}=\overline{OQ}\ and\ t_2\vec{b}=\overline{OS}$

$\therefore$By parallelogram law of vector addition,

$\overline{OR}=\overline{OQ}+\overline{OS}$

$\therefore \vec{r}=t_1\vec{a}+t_2\vec{b}$

Thus, $\vec{r}$ can be expressed as a linear combination of $\vec{a}\ and\ \vec{b}$.

Conversely, if $\vec{r}=t_1\vec{a}+t_2\vec{b}$ then clearly the vector $\vec{r}$ is in the plane determined by v.

$\therefore \vec{r}$ is coplanar with $\vec{a}\ and\ \vec{b}$

For Uniqueness:

Consider, $\vec{r} = t_1\vec{a} + t_2\vec{b}$ ....(i)

and $\vec{r} = r_1\vec{a} + r_2\vec{b}$ ....(ii)

Subtracting (ii) from (i), we get

$\vec{0} = (t_1 - r_1)\vec{a} + (t_2 - r_2)\vec{b}$

But $\vec{a}$ and $\vec{b}$ are non-collinear vectors (given)

$\therefore$ By Theorem 1,

$$t_1 - r_1 = 0 = t_2 - r_2$$

$\therefore t_1 = r_1$ and $t_2 = r_2$

Therefore the uniqueness follows.

Thus, every vector coplanar with $\vec{a}$ and $\vec{b}$ is expressed as a linear combination of $\vec{a}, \vec{b}$

In particular, in two dimensional co-ordinate system, the position vector of the point $P(x, y)$ is expressed as a linear combination of unit vectors $\hat{i}$ and $\hat{j}$ along X-axis and Y-axis, respectively.

$\therefore \vec{r} = \overline{OP} = x\hat{i} + y\hat{j}$

Theorem :13

Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if and only if there exists a non-zero linear combination $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$.

Proof

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be co-planar.

Case-1

Suppose that any two of $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear vectors, say $\vec{a}$ and $\vec{b}$.

$\therefore$ There exist scalar x, y at least one of which is non zero such that $x\vec{a} + y\vec{b} = \vec{0}$

$\therefore x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ is a required non-zero linear combination, where z=0.

Case -2

None of the two vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear

As $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$,

$\therefore$ scalars x, y are such that $\vec{c} = x\vec{a} + y\vec{b}$ …………(By theorem 2)

$\therefore x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, is a required non-zero linear combination, where z=-1

Conversely, suppose $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where one of x, y, z is non-zero, say $z \neq 0$, then

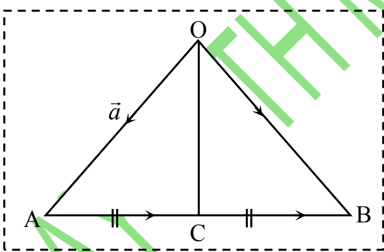
$$\vec{c} = \frac{x}{z}\vec{a} + \frac{y}{z}\vec{b}$$

$\therefore \vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$

$\therefore \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors.

Mid –point Formula

Let C be the mid-point of segment AB. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the points A, B and C $\vec{a}$ respectively.



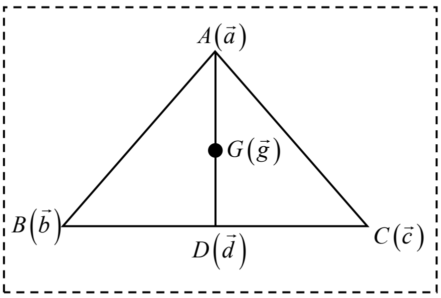
Since, C is the mid-point of segment AB, also $\overline{AC}$, $\overline{CB}$ are in the same direction

$\therefore \overline{AC} = \overline{CB}$

$\therefore$ C divides segment AB internally in the ratio 1 : 1.

$\therefore$ By section formula, $\vec{c} = \dfrac{1(\vec{b}) + 1(\vec{a})}{1+1} = \dfrac{\vec{a}+\vec{b}}{2}$

Centroid Formula of Triangle



Consider ΔABC. Let $\vec{a}, \vec{b}$ *and* $\vec{c}$ be the position vectors of the vertices A, B and C respectively, of ΔABC.

Let D be mid-point of BC.

∴ By mid-points formula, we get

$$\vec{d} = \frac{\vec{b}+\vec{c}}{2} \qquad \ldots\ldots.(i)$$

Let G be the centroid of ΔABC and $\vec{g}$ be its position vector.

∴G divides the median AD internally in the ratio 2 : 1

∴ By section formula, we get $\vec{g} = \dfrac{2(\vec{d})+1(\vec{a})}{2+1} = \dfrac{2\left(\dfrac{\vec{b}+\vec{c}}{2}\right)+\vec{a}}{3}$ …..[From (i)]

$$\therefore\ \vec{g} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Remarks:

(i) If $A \equiv (a_1, a_2, a_3)\, B \equiv (b_1, b_2, b_3)$ and $R \equiv (r_1, r_2, r_3)$ divides the segment AB in the ratio m: n, then $r_i = \dfrac{mb_i + na_i}{m+n}; i = 1,2,3$

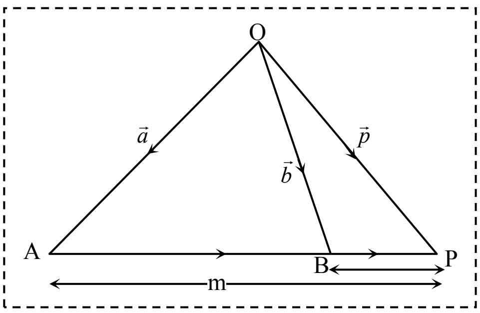
(ii) Whenever the ratio in which a point r divides segment AB is required, take the ratio to be otherwise the division in external.

Theorem 14

(External division)

Let $A(\vec{a})$ *and* $B(\vec{b})$ be any two points in the space and $P(\vec{p})$ be a third point on the line AB dividing the segment AB externally in the ratio m: n. Then $\vec{p} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$

Proof



Let AB be any line such that point p divided segment AB externally in the ratio m : n.

Let $\overline{OA}=\vec{a},\overline{OP}=\vec{p}\ and\ \overline{OB}=\vec{b}$ be the position vectors of points A, P, B respectively.

Since P divided AB externally in the ratio m : n.

$$\therefore \frac{AP}{BP}=\frac{m}{n}$$

$$\therefore n(AP)=m(BP)$$

$\overline{AP}\ and\ \overline{BP}$ are in the same direction.

$$\therefore n\left(\overrightarrow{AP}\right)=m\left(\overrightarrow{BP}\right)$$

$$\therefore n(\vec{p}-\vec{a})=m\left(\vec{p}-\vec{b}\right)$$

$$\therefore n\vec{p}-n\vec{a}=m\vec{p}-m\vec{b}\ \therefore \vec{p}(m-n)=m\vec{b}-n\vec{a}$$

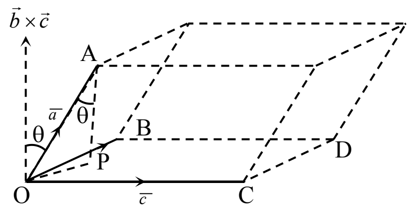
$$\therefore \vec{p}=\frac{m\vec{b}-n\vec{a}}{m-n}$$

This is the section formula for external division.

Volume of Parallelopiped

Theorem -15

The volume of a parallelepiped with coterminous edges as $\vec{a},\vec{b}\ and\ \vec{c}\ is\left[\vec{a}\vec{b}\vec{c}\right]$



Proof:

Let, $\vec{a},\vec{b}\ and\ \vec{c}$ be the position vectors of points A, B and C respectively with respect to origin O. (See the figure)

Complete the parallelepiped as shown in the figure with $\overline{OA},\overline{OB}\ and\ \overline{OC}$ as its coterminous edges.

AP is a perpendicular drawn to the plane of $\vec{b}$ and $\vec{c}$. Let, be the angle made by AP with OA.

Volume of parallelepiped

=(Area of parallelogram OCDB) ×(height)

Now, area of parallelogram $OCDB=\left|\vec{b}\times\vec{c}\right|$ …..(i)

Height of parallelepiped $= l(AP)$

$$= l(OA)\cos\theta$$

$$= \left|\overline{OA}\right|\cos\theta$$

$$= |\vec{a}|\cos\theta \qquad \text{..........(ii)}$$

$\therefore$ From (i) and (ii) we get, volume of parallelepiped $= |\vec{a}|\left|\vec{b}\times\vec{c}\right|\cos\theta$

$\therefore$ Volume of parallelepiped $= \left[\vec{a}\vec{b}\vec{c}\right]$

Theorem:16

Prove that the medians of a triangle are concurrent

Proof:

Consider Δ ABC.

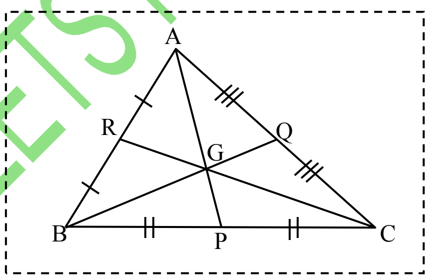
Let P, Q, R be the midpoints of the sides BC, CA, AB respectively.

Let the medians BQ and CR intersect at G.

To prove that the third median AP also passes through G.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{p}, \bar{q}, \bar{r}, \bar{g}$ be the position vectors of the points A, B, C, P, Q, R, G respectively.

Since, P, Q, R are the mid-points of the side BC, CA, AB respectively.



$\therefore$ By midpoint formula, we get

$$\left.\begin{aligned} \bar{p} &= \frac{\bar{b}+\bar{c}}{2} \quad \text{....(i)} \\ \bar{q} &= \frac{\bar{c}+\bar{a}}{2} \text{..........(ii)} \\ \bar{r} &= \frac{\bar{a}+\bar{b}}{2} \text{..........(iii)} \end{aligned}\right\}$$

From (i), (ii) and (iii), we get

$$2\overline{p} = \overline{b} + \overline{c} \Rightarrow 2\overline{p} + \overline{a} = \overline{a} + \overline{b} + \overline{c}$$

$$2\overline{q} = \overline{c} + \overline{a} \Rightarrow 2\overline{q} + \overline{b} = \overline{a} + \overline{b} + \overline{c}$$

$$2\overline{r} = \overline{a} + \overline{b} \Rightarrow 2\overline{r} + \overline{c} = \overline{a} + \overline{b} + \overline{c}$$

$$\therefore \frac{2\overline{p}+\overline{a}}{3} = \frac{2\overline{q}+\overline{b}}{3} = \frac{2\overline{r}+\overline{c}}{3} = \frac{\overline{a}+\overline{b}+\overline{c}}{3}$$

$$\therefore \frac{2\overline{p}+\overline{a}}{2+1} = \frac{2\overline{q}+\overline{b}}{2+1} = \frac{2\overline{r}+\overline{c}}{2+1} = \frac{\overline{2}+\overline{b}+\overline{c}}{3}$$

Let, $\overline{g} = \dfrac{\overline{a}+\overline{b}+\overline{c}}{3}$

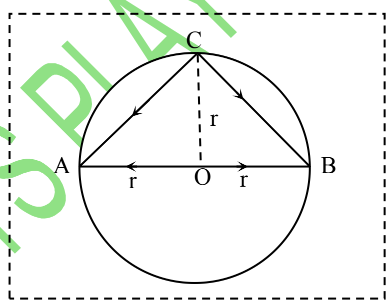
This shows that the point G whose position vector is $\overline{g}$ lies on the three medians AP, BQ, CR dividing them internally in the ratio 2 : 1.

Hence, the three medians are concurrent and meet at the point whose position vector is $\dfrac{\overline{a}+\overline{b}+\overline{c}}{3}$.

Theorem 17

Prove that the angle subtended on a semicircle is a right angle.

Proof:



Let , r be the radius and O be the centre of the circle A, B and C are three points on the circle such that, AB is the diameter.

Let $\overline{a}, \overline{b}$ *and* $\overline{c}$ be the position vectors of points A, B and C respectively.

$\therefore$ C is on the circle, $|\overline{c}| = r$.

Also, $|\overline{a}| = |\overline{b}| = r$ and $\overline{b} = -\overline{a}$

Consider $\overline{CA}.\overline{CB} = (\overline{a} - \overline{c}).(\overline{b} - \overline{c})$

$$= (\overline{a} - \overline{c}).(-\overline{a} - \overline{c})$$

$=\left(\bar{a}-\bar{c}\right).\left(-1\right)\left(-\bar{a}-\bar{c}\right)$

$=\left(-1\right)\left(\bar{a}-\bar{c}\right)\left(\bar{a}-\bar{c}\right)$

$=\left(\left|\bar{c}\right|^2-\left|\bar{a}\right|^2\right)$

$=r^2-r^2$

$=0$

$\therefore \overline{CA}.\overline{CB}=0$

$\therefore \overline{CA}$ is perpendicular to $\overline{CB}$

$\therefore$ The angle between $\overline{CA}$ *and* $\overline{CB}$ is a right angle

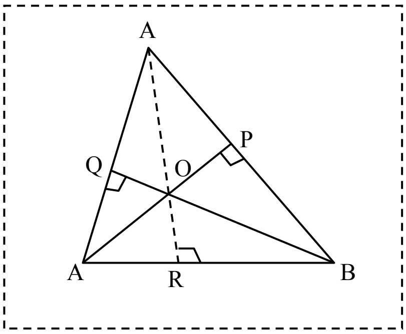
$\therefore m\angle ACB=90^o$

$\therefore$ The angle subtended on a semicircle is a right angle.

Theorem:18

Prove that the altitudes of a triangle are concurrent

Proof:



Consider ΔABC.

Let AP⊥BC and BQ ⊥ AC.

Let AP and BQ intersect at O. Join OC and extend OC to meet AB at R. To prove that CR is also the altitude of ΔABC.

i.e. to prove that CR ⊥ AB

Consider $\overline{AP}\perp\overline{BC}$

$\therefore \overline{AO}\perp\overline{BC}$

$\therefore \overline{AO}.\overline{BC}=0$

$\therefore -\bar{a}.\left(\bar{c}-\bar{b}\right)=0$ ...... $\left[\because \overline{AO}=-\overline{OA}\right]$

$\therefore \bar{a}.\bar{c}-\bar{a}.\bar{b}=0$ ....(i)

Now, $\overline{BQ}\perp\overline{AC}$

$\therefore \overline{BO}\perp\overline{AC}$

$\therefore \overline{BO}.\overline{AC} = 0$

$\therefore -\overline{b}.(\overline{c} - \overline{a}) = 0$ ....$[\because \overline{BO} = -\overline{OB}]$

$\therefore \overline{b}.\overline{c} - \overline{b}.\overline{a} = 0$ ....(ii)

Comparing equations (i) and (ii), we get

$\overline{a}.\overline{c} - \overline{a}.\overline{b} = \overline{b}.\overline{c} - \overline{b}.\overline{a}$

$\therefore \overline{a}.\overline{c} = \overline{b}.\overline{c}$

$\therefore \overline{a}.\overline{c} - \overline{b}.\overline{c} = 0$

$\therefore \overline{c}.(\overline{a} - \overline{b}) = 0$

$\therefore -\overline{c}.(\overline{a} - \overline{b}) = 0$

$\therefore \overline{CO} \perp \overline{BA}$

$\therefore \overline{CR} \perp \overline{BA}$

$\therefore CR \perp BA$

$\therefore$ CR is also altitude of ΔABC.

$\therefore$ AP, BQ, CR intersect at O.

$\therefore$ All three altitude of ΔABC intersect at a common point.

Thus, the altitudes of a triangle are concurrent.

Theorem 19

If $y = f(x)$ is a differentiable function of x such that inverse function $x = f^{-1}(y)$ exists, then x is a differentiable function of y and $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$, where $\frac{dy}{dx} \neq 0$.

Proof.

'y' is a differentiable function of 'x'.

Let there be a small change δx in the value of 'x'.

Correspondingly, there should be a small change δx in the value of 'y'.

As $\delta x \to 0, \delta y \to 0$

Consider, $\frac{\delta x}{\delta y}=\frac{1}{\frac{\delta y}{\delta x}}, \frac{\delta y}{\delta x}\neq 0$

Taking $\lim_{\delta x\to 0}\left(\frac{\delta x}{\delta y}\right)=\frac{1}{\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)}$

Since 'y' is a differentiable function of 'x'.

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)=\frac{dy}{dx}$

As $\delta x\to 0, \delta y\to 0$

$$\lim_{\delta x\to 0}\left(\frac{\delta x}{\delta y}\right)=\frac{1}{\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)} \qquad \text{..........(i)}$$

$\therefore$ limits on R.H.S of (i)exist and are finite. Hence, limits on L.H.S of (i) also should exist and be finite.

$\therefore \lim_{\delta y\to 0}\left(\frac{\delta x}{\delta y}\right)=\frac{dx}{dy}$ exists and is finite.

$\therefore \frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}, \frac{dy}{dx}\neq 0$

Theorem20

If $x=f(t)$ and $y=g(t)$ are two differentiable functions of parameter t such that y is a function of x, then prove that

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right), \frac{dx}{dt}\neq 0,$$

Proof

x and y are differentiable function of t.

Let there be a small increment δt in the value of t. Correspondingly, there should be a small increments $\delta x, \delta y$ in the values of x and y respectively.

As $\delta t\to 0, \delta x\to 0, \delta y\to 0$

Consider, $\dfrac{\delta y}{\delta x}=\dfrac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}$

Taking $\lim\limits_{\delta t\to 0}$ on both sides, we get

$$\lim_{\delta t\to 0}\frac{\delta y}{\delta x}=\frac{\lim\limits_{\delta t\to 0}\frac{\delta y}{\delta t}}{\lim\limits_{\delta t\to 0}\frac{\delta x}{\delta t}}$$

Since x and y are differentiable functions of t, $\lim\limits_{\delta t\to 0}\frac{\delta y}{\delta t}=\frac{dy}{dt}$ exists and is finite.

$\therefore \lim\limits_{\delta t\to 0}\frac{\delta x}{\delta t}=\frac{dx}{dt}$ exists and is finite.

$$\therefore \lim_{\delta t\to 0}\frac{\delta y}{\delta x}=\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$\therefore \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)(as\ \delta t\to 0, \delta x\to 0)$$

$\therefore \lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{dy}{dx}$ exists and is finite.

$$\therefore \frac{dy}{dx}=\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right), \frac{dx}{dt}\neq 0$$

Theorem 21

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$

Proof

Let $I=\int\sqrt{a^2-x^2}\,.1dx$

$$=\sqrt{a^2-x^2}\int 1dx-\int\left[\frac{d}{dx}\left(\sqrt{a^2-x^2}\right).\int 1dx\right]dx$$

$$= x.\sqrt{a^2 - x^2} - \int \frac{-2x}{2\sqrt{a^2 - x^2}}.\text{xdx}$$

$$= x.\sqrt{a^2 - x^2} - \int \frac{(a^2 - x^2) - a^2}{\sqrt{a^2 - x^2}} dx$$

$$= x.\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} - \frac{a^2}{\sqrt{a^2 - x^2}} \right) dx$$

$$= x.\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx$$

$$\therefore I = x.\sqrt{a^2 - x^2} - I + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore 2I = x.\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c_1$$

$$\therefore I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{c_1}{2}$$

$$\therefore \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$$

Where $c = \frac{c_1}{2}$

Theorem 22

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left| x + \sqrt{x^2 + a^2} \right| + c$$

Proof

Let $I = \int \sqrt{x^2 + a^2}\, .1dx$

$$= \sqrt{x^2 + a^2} \int 1dx - \int \left[\frac{d}{dx}\left(\sqrt{x^2 + a^2}\right). \int 1dx \right] dx$$

$$= x.\sqrt{x^2 + a^2} - \int \frac{2x}{2\sqrt{x^2 + a^2}}.xdx$$

$$= x.\sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}} dx$$

$$= x.\sqrt{x^2+a^2} - \int\left(\frac{x^2+a^2}{\sqrt{x^2+a^2}} - \frac{a^2}{\sqrt{x^2+a^2}}\right)dx$$

$$= x.\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2\int\frac{1}{\sqrt{x^2+a^2}}dx$$

$$\therefore I = x.\sqrt{x^2+a^2} - I + a^2\log\left|x+\sqrt{x^2+a^2}\right| + c_1$$

$$\therefore 2I = x.\sqrt{x^2+a^2} + a^2\log\left|x+\sqrt{x^2+a^2}\right| + c_1$$

$$\therefore I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + \frac{c_1}{2}$$

$$\therefore \int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$$

Where $c = \frac{c_1}{2}$

Theorem 23

$$\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$$

Proof

Let $I = \int\sqrt{x^2-a^2}.1dx$

$$= \sqrt{x^2-a^2}\int 1dx - \int\left[\frac{d}{dx}\left(\sqrt{x^2-a^2}\right).\int 1dx\right]dx$$

$$= x.\sqrt{x^2-a^2} - \int\frac{2x}{2\sqrt{x^2-a^2}}.\text{xdx}$$

$$= x.\sqrt{x^2-a^2} - \int\frac{\left(x^2-a^2\right)+a^2}{\sqrt{x^2-a^2}}dx$$

$$= x.\sqrt{x^2-a^2} - \int\left(\frac{x^2-a^2}{\sqrt{x^2-a^2}} + \frac{a^2}{\sqrt{x^2-a^2}}\right)dx$$

$$= x.\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\int\frac{1}{\sqrt{x^2-a^2}}dx$$

$$\therefore I = x.\sqrt{x^2-a^2} - I - a^2\log\left|x+\sqrt{x^2-a^2}\right| + c_1$$

$$\therefore 2I = x.\sqrt{x^2 - a^2} - a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c_1$$

$$\therefore I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + \frac{c_1}{2}$$

$$\therefore \int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$$

Where $c = \frac{c_1}{2}$

Theorem: 24

Prove that $\int_a^b f(x)\,dx = \int_a^e f(x)\,dx + \int_c^b f(x)\,dx,$ **where** $a < c < b$

Proof

Let $\int f(x)\,dx = F(x)$

Consider $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$

$= F(b) - F(a)$ ……….(i)

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \left[F(x)\right]_a^c + \left[F(x)\right]_c^b$$

$= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$ ……..(ii)

From (i) and (ii), we get

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, where\, a < c < b$$

Theorem:25

Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof

Let $I = \int_0^a f(x)\,dx$

Put $x = a - t$

$\therefore dx = -dt$

When $x = 0, t = a - 0 = a$

When $x = a, t = a - a = 0$

$$\therefore I = \int_0^a f(x)\,dx = \int_a^0 f(a-t)(-dt)$$

$$= -\int_a^0 f(a-t)\,dt$$

$$= \int_0^a f(a-t)\,dt$$

$$\ldots\ldots\ldots\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \int_0^a f(a-x)\,dx$$

$$\ldots\ldots\ldots\left[\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right]$$

$$\therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Theorem:26

Prove that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Proof

Let $I = \int_a^b f(x)dx$

Put $x = a+b-t$

$\therefore dx = -dt$

When $x = a, t = b$ and when $x = b, t = a$

$\therefore I = \int_a^b f(x)dx = \int_b^a f(a+b-t)(-dt)$

$= -\int_b^a f(a+b-t)dt$

$= \int_a^b f(a+b-t)dt$

$\ldots\ldots \left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$

$= \int_a^b f(a+b-x)dx$

$\ldots\ldots\ldots \left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$\therefore \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Theorem:27

Prove that $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Proof:

Let $I = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$= I_1 + I_2$

From I₁, put $2a - x = t$

$\therefore -dx = dt \therefore dx = -dt$

 When $x = 0, t = 2a$ and when $x = a, t = a$

$$\therefore I_2 = \int_0^a f(2a-x)\,dx = -\int_{2a}^{a} f(t)\,dt$$

$$\therefore I = I_1 + I_2$$

$$= \int_0^a f(x)\,dx - \int_{2a}^{a} f(t)\,dt$$

$$= \int_0^a f(x)\,dx + \int_a^{2a} f(t)\,dt$$

$$\ldots\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

$$\ldots\left[\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right]$$

$$= \int_0^{2a} f(x)\,dx$$

$$\ldots\ldots\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx; a < c < b\right]$$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Theorem:28

$\int_{-a}^{a} f(x)\,dx = 2.\int_0^a f(x)\,dx,$ if $f(x)$ is an even function $= 0$,

Proof:

L.H.S becomes $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

$$\int_{-a}^{a} f(x)\,dx = I + \int_{0}^{a} f(x)\,dx \qquad \ldots\ldots\ldots\text{(i)}$$

Now, $I = \int_{-a}^{0} f(x)\,dx$

Put $x = -t$

$\therefore dx = -dt$

When $x = -a, t = a$ and when $x = 0, t = 0$

$$\therefore I = \int_{a}^{0} f(-t)(-dt) = -\int_{a}^{0} f(-t)\,dt$$

$$= \int_{0}^{a} f(-t)\,dt \quad \ldots\ldots\ldots \left[\because \int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx\right]$$

$$= \int_{0}^{a} f(-x)\,dx \qquad \ldots\ldots\ldots \left[\because \int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(t)\,dt\right]$$

$\therefore$Equation (i) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} \left[f(-x) + f(x)\right]dx \qquad \ldots\ldots\ldots\ldots\text{(ii)}$$

Case 1: If $f(x)$ is an even function, then $f(-x) = f(x)$,

Thus, equation (ii) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \left[f(x) + f(x)\right]dx = 2.\int_{0}^{a} f(x)\,dx$$

Case 2: If $f(x)$ is an odd function, then $f(-x) = -f(x)$, Thus, equation (ii) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \left[-f(x) + f(x)\right]dx = 0$$